

ON THE INCLUSION $D_5(G) \subseteq \gamma_4(G)$: REDUCTIONS, THE LIE-RING PROOF, AND EVIDENCE

ABSTRACT. For a group G let $D_n(G) = G \cap (1 + \Delta(G)^n)$ be its integral dimension subgroups and $\gamma_n(G)$ its lower central series. Whether $D_5(G) \subseteq \gamma_4(G)$ for every group is open (Kourovka Notebook, Problem 20.84). For Lie rings the analogous inclusion was recently proved by Ionin, Mikhailov and Petrov, and formally verified in Lean. This note collects what is known, records an elementary reduction of the group question to finite metabelian 2-groups of class three, analyses which steps of the Lie-ring proof have no group-theoretic counterpart, and reports computational evidence. For 3-torsion in D_5/γ_5 we prove $D_5(G) = \gamma_5(G)$ whenever $G/\gamma_2(G)$ is finite and has at most one invariant larger than 3, and, with a computer search, for all 3-groups of order at most 3^9 . *We do not solve the problem.*

1. THE QUESTION

Let $\Delta = \Delta(G)$ be the augmentation ideal of $\mathbb{Z}[G]$ and $D_n(G) = \{g \in G : g - 1 \in \Delta^n\}$. Always $\gamma_n(G) \subseteq D_n(G)$ **[paper; Lean: §4]**, with equality for $n \leq 3$ and failure for $n = 4$ in general (Rips [Rip72]).

Question 1.1 ([KM26, Problem 20.84], [IMP26, §1.3]). *Is $D_5(G) \subseteq \gamma_4(G)$ for every group G ? **[open]***

Known facts used below (all **[paper]**, from the cited sources): $D_4(G)^2 \subseteq \gamma_4(G)$ (Passi [Pas68], see also [MP09]); $D_n(G)/\gamma_n(G)$ is abelian [GK92]; the exponent of $D_5(G)/\gamma_5(G)$ divides $3! = 6$ (Tahara [Tah81, Cor. 6.11], proved there for finite groups, to which the general case reduces by the standard argument; Tahara also gives an explicit description of D_5 for finite groups of class 4 [Tah81, Thm. 6.1] and shows $D_k = \gamma_k$ for $k \leq 5$ when G has no elements of order 2 or 3 [Tah81, Cor. 6.12]); Gupta's claim that all D_n/γ_n are 2-groups [Gup02] is false: Bartholdi and Mikhailov constructed a 3-group with $D_7 \neq \gamma_7$ [BM23]. See also our analysis of Gupta's argument (wiki, MATH-3).

2. AN ELEMENTARY REDUCTION

Proposition 2.1. *The following are equivalent.*

- (1) $D_5(G) \subseteq \gamma_4(G)$ for every group G .

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- (2) $D_5(M) = 1$ for every metabelian group M of nilpotency class at most 3.
- (3) $D_5(Q) = 1$ for every finite metabelian 2-group Q of nilpotency class exactly 3.

Proof **[paper; Lean]** (in Lean with the theorems of Passi and Gruenberg and $D_3 = \gamma_3$ as explicit hypothesis)

(1) $\Rightarrow$ (3) is clear since $\gamma_4(Q) = 1$.

(2) $\Rightarrow$ (1). Put $N = \gamma_4(G)G''$. Since $G'' = [\gamma_2(G), \gamma_2(G)] \subseteq \gamma_4(G)$, $N = \gamma_4(G)$, and $M = G/N$ is metabelian of class ≤ 3 . The projection $\pi: G \rightarrow M$ induces a ring map $\mathbb{Z}[G] \rightarrow \mathbb{Z}[M]$ sending $\Delta(G)^n$ into $\Delta(M)^n$, so $\pi(D_5(G)) \subseteq D_5(M) = 1$, i.e. $D_5(G) \subseteq N = \gamma_4(G)$. (This is the group version of the step from Theorem C to Corollary D in [IMP26].)

(3) $\Rightarrow$ (2). Let M be metabelian of class ≤ 3 and $1 \neq x \in D_5(M)$. *Finite generation.* $x - 1$ is a finite $\mathbb{Z}$ -linear combination of products $(g_1 - 1) \cdots (g_5 - 1)$; all g_i and x lie in a finitely generated subgroup H , and then $x \in D_5(H)$. H is metabelian of class ≤ 3 , hence finitely generated nilpotent. *Exponent two.* By Passi, $D_4(H)^2 \subseteq \gamma_4(H) = 1$; as $D_5 \subseteq D_4$, $x^2 = 1$. *Finite 2-quotients.* Let T be the (finite, normal) subgroup of elements of odd order of H ; $x \notin T$. The group H/T is finitely generated nilpotent with 2-group torsion, hence residually a finite 2-group (Gruenberg [Gru57]). Choose a finite 2-group quotient Q of H in which the image of x is nontrivial; by functoriality it lies in $D_5(Q)$. Q is metabelian of class ≤ 3 , and class ≤ 2 is impossible because then $D_5(Q) \subseteq D_3(Q) = \gamma_3(Q) = 1$. $\square$

Remark 2.2 (“metabelian” is automatic). Every group of nilpotency class ≤ 3 is metabelian: $G'' = [\gamma_2(G), \gamma_2(G)] \subseteq \gamma_4(G)$ by P. Hall’s inclusion $[\gamma_i, \gamma_j] \subseteq \gamma_{i+j}$ (see e.g. [MP09]); this is the same fact used in (2) $\Rightarrow$ (1). So in (2) and (3) the word “metabelian” may be dropped: (3) is the statement that $D_5(Q) = 1$ for every finite 2-group of class 3, and the search space of §5 is simply the 2-groups of class 3. The same holds for Lie rings ($[\gamma_2, \gamma_2] \subseteq \gamma_4$), so Theorem C of [IMP26] for $c = 3$ needs no metabelian hypothesis either; the hypothesis matters only for $c \geq 4$, where $D_{2c-1} = 1$ for all groups of class $\leq c$ is a different, stronger question. This standard fact was noticed again only while setting up the ANUPQ search (MATH-37), where the option **Metabelian** cut nothing. **[paper]**

Remark 2.3 (a bounded form). For finite groups the proof gives more. If G is finite then so are $M = G/\gamma_4(G)$ and $H \leq M$, and one may take $Q = H/T$, a finite 2-group whose order is the 2-part of $|H|$, hence divides the 2-part of $|G|$. Therefore: *if $D_5(Q) = 1$ for all metabelian 2-groups Q of class 3 and order $\leq 2^k$, then $D_5(G) \subseteq \gamma_4(G)$ for every finite group G whose Sylow 2-subgroup has order $\leq 2^k$* **[paper; Lean]** (in Lean Gruenberg’s theorem is not needed: there $H = M$, and Q is the Sylow 2-subgroup of M , a direct factor since M is nilpotent; for Q of class ≤ 3 Passi’s theorem is the only hypothesis, for class exactly 3 also $D_3 = \gamma_3$). Note that the exponent step uses Passi’s theorem for H (where $\gamma_4(H) = 1$), not for an arbitrary group.

Remark 2.4. Proposition 2.1(2) is precisely the group analogue of Theorem C of [IMP26] for $c = 3$ (“a metabelian Lie ring of class $\leq c$ has $D_{2c-1} = 0$ ”). The reduction is elementary and almost certainly known to experts; we record it because it fixes the search space for computations (§5).

3. THE LIE-RING PROOF AND WHERE IT DOES NOT TRANSFER

We summarise the proof of [IMP26, Theorem C, Corollary D], checked against the text ([paper]; the final statements are [Lean], audited in our wiki, MATH-6).

- (1) Reduction to L metabelian of class $\leq c$ (Corollary 6.14) and to finitely generated L (Lemma 6.13). *Transfers*: this is Proposition 2.1.
- (2) A central extension $0 \rightarrow K \rightarrow T \rightarrow L \rightarrow 0$, $T = L \times_{L_{ab}} P$ with P free abelian, such that $\gamma_i(T) \cong \gamma_i(L)$ for $i \geq 2$ and $T \cong T' \oplus P$ additively (Lemma 6.4, Corollary 6.7).
- (3) The integral PBW theorem: $\text{gr } U(T) \cong \text{Sym}_{\mathbb{Z}}(T)$ and, for T a direct sum of cyclic groups, an explicit $\mathbb{Z}$ -basis of ordered monomials with known additive orders (Theorem 2.3, Corollary 2.4).
- (4) The additive projection $p: U(T) \rightarrow P_{\leq 2}U(T)$ onto monomials of length ≤ 2 (§6.2), and a retraction $r: P_{\leq 2}U(T) \rightarrow T$ killing $Z(T) \cdot T$ (Lemma 5.3).
- (5) The quadratic part of $p(\varpi(T)^n)$ lies in $\sum_{i+j=n} \gamma_i(T)\gamma_j(T)$ (Proposition 6.12); for $n = 2c - 1$ every such term has a central factor, and Theorem 5.1 gives $D_{2c-1}(L) = 0$.

Obstacles for groups (our analysis; [paper] where a source is cited, otherwise an explanation, not a theorem):

- *No PBW basis of the required kind.* Step 3 is the heart of the argument: it gives an additive splitting of $U(T)$ by monomial length, compatible with additive orders, so that “the quadratic part” of an element is well defined and computable. For $\mathbb{Z}[G]$ the filtration by powers of Δ has associated graded ring $\bigoplus \Delta^n / \Delta^{n+1}$, which in general is *not* the integral enveloping algebra of $\text{gr } \gamma_{\bullet}(G)$; over $\mathbb{Q}$ the analogous comparison is Quillen’s theorem, and over $\mathbb{Z}$ its failure is essentially the dimension subgroup problem itself (cf. [HMP07, MP09]). There is no canonical additive projection of $\mathbb{Z}[G]$ onto “length ≤ 2 ”.
- *The extension T .* Step 2 has a formal group analogue (pull back a free abelian cover of G_{ab} along $G \rightarrow G_{ab}$, giving a central extension with the same higher lower central terms). What is missing is how to use it: in the Lie case T' is an additive direct summand and its elements commute in $U(T)$, which drives the rewriting Lemmas 6.9–6.10. In $\mathbb{Z}[G]$ products of commutators do not commute and the corresponding rewriting produces higher-order terms (“collection”).
- *Torsion is already there in degree four.* For Lie rings $D_n = \gamma_n$ for $n \leq 3$ and $2D_4 \subseteq \gamma_4$ [BP15], the same pattern as for groups; but for groups even the description of D_4/γ_4 requires the full Rips–Passi

machinery, and the known group computations of D_5 (e.g. the group of [IMP26, §3.3], where $D_5(G) = \langle w \rangle \subseteq \gamma_4(G)$) are case-by-case.

4. FORMALIZATION

The Lean 4 project `groups/` (Mathlib v4.29.0, the version used by [IMP26]) defines $\Delta(G)$ as the kernel of the augmentation of `MonoidAlgebra  $\mathbb{Z}$  G` and $D_n(G) = \{g : g-1 \in \Delta^n\}$, with literature indexing; Mathlib’s `lowerCentralSeries` `G k` is γ_{k+1} . The following are proved, with no `sorry` and only the standard axioms `propext`, `Classical.choice`, `Quot.sound` [Lean]:

- $D_n(G)$ is a normal subgroup, $D_0 = D_1 = G$, the series is decreasing, and $\Delta^i \Delta^j \subseteq \Delta^{i+j}$;
- the commutator identities $[a, b]-1 = a^{-1}b^{-1}((a-1)(b-1)-(b-1)(a-1))$ (for $[a, b] = a^{-1}b^{-1}ab$) and its analogue for Mathlib’s convention; consequently $[D_i, D_j] \subseteq D_{i+j}$ and $\gamma_n(G) \subseteq D_n(G)$;
- functoriality of D_n under group homomorphisms;
- $D_2(G) = \gamma_2(G)$, via a ring map $\mathbb{Z}[G] \rightarrow \mathbb{Z} \oplus G_{ab}$ (square-zero extension) killing Δ^2 ;
- Proposition 2.1 and Remark 2.3: first, $D_5(G) \subseteq \gamma_4(G)$ for all G iff $D_5(Q) = 1$ for all Q of class ≤ 3 ; then, with Passi’s $D_4^2 \subseteq \gamma_4$ and Gruenberg’s theorem as explicit hypotheses, iff $D_5(Q) = 1$ for all finite 2-groups of class ≤ 3 , and with $D_3 = \gamma_3$ also of class exactly 3; the bounded form for finite G does not need Gruenberg’s theorem; “metabelian” is dropped by Remark 2.2, which is not formalized (`dimensionQuestion5_iff_finite_two_groups`, `dimensionQuestion5_iff_class_three`, `D_five_le_of_bounded`, `D_five_le_of_bounded_class` file `FiniteReduction.lean`). Finite generation is proved: every element of $\Delta(G)^n$ comes from $\Delta(H)^n$ for a finitely generated $H \leq G$;
- Lemma 5.9 (adjoining a central root): $\mathbb{Z}[H] \cap \Delta(K)^n = \Delta(H)^n$ and $D_n(K) = D_n(H)$ for $n \geq 2$ (`D_n_eq_of_index`, file `CentralRoot.lean`);
- Proposition 5.8, all cases, with the ring statement $\mathbb{Z}[H] \cap \Delta(Q)^n = \Delta(H)^n$, for $Q = H\langle a, b \rangle$ with a, b centralising H and $a, b, ab \notin H$; also the step of adjoining a central square root in (2) and the variant $c^{2^j} \in \gamma_{j+2}(H)$ (`D_n_eq_of_dihedral`, `D_n_eq_of_sqrt`, `D_n_eq_of_central_product` and its corollaries; files `CentralD8.lean`, `CentralSqrt.lean`, `CentralProduct.lean`). The passage from $(H \times E)/\{(\varphi(z), z^{-1})\}$ to this form and the classification of E are not formalized. The formal proof uses the same overgroup $(Q \times \mathbb{Z})/\langle (s, -2) \rangle$ for Q_8 and $M_{2^{k+2}}$, with $s = a^2$ ($s = t$ for Q_8) and $b^2 = s^e$ ($e = 1$ for Q_8 , $e = 0$ for $M_{2^{k+2}}$): the image ζ of the generator of $\mathbb{Z}$ has $\zeta^2 = s$, and $a\zeta^{-1}$, $b\zeta^{-e}$ generate a subgroup D_8 centralising $H\langle \zeta \rangle$;
- Lemma 6.3 and Proposition 6.5, with $d(Q)$ the least number of generators (`commutator_gamma_two_le_gamma_five`, `metabelian_of_class_four`, `card_ge_of_not_metabelian`, `metabelian_of_card_le`, `rank_of_card_nine`, `type_of_card_nine`; files `ClassFour.lean`, `ClassFourOrder.lean`);

in the proposition the class is assumed ≤ 4 instead of $= 4$. Instead of the Lie ring $L(G)$ the proof works in $G/\gamma_5(G)$, where γ_4 is central: the brackets become $F([a, b])$ for an additive $F: \gamma_4 \rightarrow \mathbb{F}_p$, nonzero at a given commutator (it exists since $\gamma_4(G/\gamma_5)$ has exponent p), and both Jacobi identities come from the Hall–Witt identity. Also proved on the way: $[\gamma_i, \gamma_j] \subseteq \gamma_{i+j}$, a group of order p^n is generated by n elements, and in a finite p -group $H\gamma_2 = G$ implies $H = G$. That both types occur for $p = 3$ is a computation and is not formalized;

- Theorems 6.6 and 6.8 modulo Tahara’s Theorem 6.1 [Tah81], which enters as an explicit hypothesis (`TaharaFiveOne`, file `Tahara.lean`): its uniqueness bases, structure constants, conditions (A)–(F) and generators are written out as in [Tah81, pp. 143, 150, 159–161], and only the inclusion of D_5 in the subgroup generated by the elements g is assumed (`D_five_eq_gamma_five`, `D_five_eq_gamma_five_cyclic`; files `ExpThree.lean`, `ExpThreePlus.lean`). In Theorem 6.8 the hypothesis reads “ G/γ_2 is a finite 3-group whose cubes form a cyclic group”, which for a finite abelian 3-group means at most one invariant > 3 . The reduction to the finite group G/γ_5 is proved too. Elements of γ_3 and γ_4 are compared through additive maps to $\mathbb{F}_3$, and the uniqueness bases adapted to the cube map come from bases of $\mathbb{F}_3$ -spaces;
- the Lie-ring inclusion $D_5(L) \subseteq \gamma_4(L)$, derived from the authors’ `LieRings.Publication.corollaryD` [IMP26].

Stated as named propositions with references, *not* proved: Question 1.1; $D_3 = \gamma_3$; $D_4^2 \subseteq \gamma_4$; Rips’s example; Gruenberg’s theorem [Gru57] in the form used in Proposition 2.1; Tahara’s bound; the two Gupta–Kuz’min statements of [GK92]; $D_5^2 \subseteq G''\gamma_5$ [MP09, Thm. 2.28]; and the group analogues of Theorem C and Corollary D of [IMP26] (without any claim about their truth). Some of them are explicit hypotheses of theorems (as in Proposition 2.1); none is assumed globally.

5. COMPUTATIONS

All code, raw tables and the checks are in `compute/` of the repository <https://github.com/ispace/dimension-subrings> (`README.md`, `results/SUMMARY.md`).

Method. For finite G , Δ is free abelian with basis $\{g - 1\}_{g \neq 1}$, and $\Delta^n = \text{span}_{\mathbb{Z}}\{b(x_i - 1) : b \text{ in a basis of } \Delta^{n-1}, x_i \text{ generators}\}$: no closure under further multiplication is needed, since Δ is the left ideal generated by the $x_i - 1$ and $\Delta^{n-1}\mathbb{Z}[G] = \Delta^{n-1}$. The lattices are computed in Howell form over $\mathbb{Z}/p^E$ ($E = 30, 19, 13$ for $p = 2, 3, 5$), which is exact for our purpose: $g - 1 \notin \Delta^n + p^E\Delta$ implies $g - 1 \notin \Delta^n$, and the equalities $D_n = \gamma_n$ below are proved by such non-memberships (memberships are certified separately by $p^{E-1}\Delta \subseteq \Delta^n + p^E\Delta$). Groups are taken from the SmallGroups library of GAP; each group is computed until $D_n = 1$.

Checks. $\gamma_n \subseteq D_n$ at every level of every group; a second method (left multiplication on conjugacy-class representatives of a pc-generating set) on all groups of orders 8–64, 81, 243, 625; an independent exact Hermite normal form over $\mathbb{Z}$ in PARI/GP on all groups of orders 8–64, 81 and 12 groups of order 128; known values ($\Delta^n/\Delta^{n+1} \cong G$ for cyclic G , $\Delta/\Delta^2 \cong G_{ab}$). No discrepancies.

Sieves (theorems, used to skip groups): p -class $\geq$ class, so groups of p -class 2 have class ≤ 2 ; for 2-groups of class 3, $D_5 \subseteq D_3 = \gamma_3$, Jennings' theorem $D_n \subseteq M_n$ (the mod-2 dimension subgroups) and Passi's $D_4^2 \subseteq \gamma_4 = 1$ give $D_5 \subseteq \Omega_1(\gamma_3 \cap M_5)$, so only groups with $K = \Omega_1(\gamma_3 \cap M_5) \neq 1$ need the lattice; for 3-groups of class 4 the analogous $K = \Omega_1(\gamma_4 \cap M_5)$, using $D_4 = \gamma_4$ for groups without 2-torsion (from Passi's $D_4^2 \subseteq \gamma_4$ [Pas68]) and Tahara's bound $D_5^6 \subseteq \gamma_5$ [Tah81, Cor. 6.11] (both **[paper]**).

Proposition 5.1 ([computer]). *In each of the following groups $D_n(G) = \gamma_n(G)$ for all n :*

- (1) *all 2-groups of order 8, 16, 32, 64, 128 (2665 groups) and all groups of order 256 of class ≥ 3 (24 328 groups);*
- (2) *all 3-groups of order 81, 243, 729 (586 groups);*
- (3) *all 1202 groups of order 3^7 of class 4 that pass the sieve (by two methods, the lattice of §5 and the truncated Magnus algebra of the next paragraph, with identical results);*
- (4) *all 15 groups of order 625 (a control).*

Every 3-group of class 4 and order $3^5, 3^6, 3^7$ (6, 71, 1309 groups) is metabelian (for this there is now a proof, Proposition 6.5, also in Lean). Moreover, $D_5(Q) = 1$ for every metabelian group Q of order 512 and class 3: the sieve leaves 2792 of the 1 587 389 such groups (out of 1 708 439 groups of order 512 and p -class ≥ 3), and for each of the 2792 the lattice computation gives $D_4(Q) = D_5(Q) = 1$.

Order 2^{10} . This order is not in SmallGroups, and generating all 2-groups of class 3 of this order is out of reach: at rank ≥ 4 the p -group generation tree has millions of nodes already at order 2^9 . Instead, the complete result at order 2^9 cuts the search down to a few thousand candidates.

Notation. $P_1(Q) = Q$, $P_{i+1}(Q) = [P_i(Q), Q] P_i(Q)^2$ is the lower exponent-2 central series, and the 2-class of Q is the least c with $P_{c+1}(Q) = 1$; $M_n(Q) = Q \cap (1 + \Delta_{\mathbb{F}_2}(Q)^n)$ are the mod-2 dimension subgroups, $M_n = \prod_{i \cdot 2^j \geq n} \gamma_i^{2^j}$ (Jennings [Jen41], Lazard). If Q has 2-class $c \geq 2$, it is an *immediate descendant* of $Q/P_c(Q)$ of step size $\log_2 |P_c(Q)|$; the p -group generation algorithm [OBr90], as implemented in ANUPQ [GNO], lists all immediate descendants of a given group of a given step size, up to isomorphism.

Lemma 5.2 (minimal counterexample). *Let Q be a finite 2-group of class 3 such that $D_5(Q/N) = 1$ for every central subgroup N of order 2. If $D_5(Q) \neq 1$, then*

- (a) *$Z(Q)$ is cyclic and $D_5(Q) = \Omega_1(Z(Q)) = \langle n \rangle$ has order 2;*

- (b) $n \in \gamma_3(Q) \cap M_5(Q)$;
- (c) the 2-class c of Q is at least 4, and $P_c(Q) = \langle n \rangle$;
- (d) Q is an immediate descendant of step size 1 of $R = Q/\langle n \rangle$, a group of class 2 or 3 and 2-class $c - 1 \geq 3$ with $\gamma_3(R)$ cyclic.

Proof [paper]. (a) Let $N \leq Z(Q)$, $|N| = 2$. The projection $Q \rightarrow Q/N$ maps $D_5(Q)$ into $D_5(Q/N) = 1$, so $D_5(Q) \subseteq N$. Hence a nontrivial $D_5(Q)$ equals *every* central subgroup of order 2: there is only one, so $\Omega_1(Z(Q))$ has order 2, the abelian 2-group $Z(Q)$ is cyclic, and $D_5(Q) = \Omega_1(Z(Q))$. (b) $D_5 \subseteq D_3 = \gamma_3$, and $D_5(Q) \subseteq M_5(Q)$ because $\mathbb{Z}[Q] \rightarrow \mathbb{F}_2[Q]$ maps Δ^5 into $\Delta_{\mathbb{F}_2}^5$. (c) Since $\gamma_i \subseteq P_i$ and $P_i^2 \subseteq P_{i+1}$, we have $\gamma_i^{2^j} \subseteq P_{i+j}$, and $i2^j \geq 5$ implies $i + j \geq 4$. So $M_5 \subseteq P_4$, and $1 \neq n \in P_4(Q)$ gives $c \geq 4$. The subgroup $P_c(Q)$ is nontrivial, central and elementary abelian, hence equal to $\Omega_1(Z(Q)) = \langle n \rangle$. (d) By (c), $R = Q/P_c(Q)$, the step size is 1 and R has 2-class $c - 1 \geq 3$. R has class ≤ 3 ; if R were abelian, then $\gamma_2(Q) \subseteq \langle n \rangle \subseteq Z(Q)$ and Q would have class ≤ 2 . Finally $\gamma_3(R)$ is a quotient of $\gamma_3(Q) \subseteq Z(Q)$, which is cyclic. $\square$

Proposition 5.3 ([computer + paper]). $D_5(Q) = 1$ for every 2-group Q of order 2^{10} and class 3 (all of them are metabelian, Remark 2.2).

Proof. Let Q be such a group with $D_5(Q) \neq 1$. For every central N of order 2 the quotient Q/N has order 2^9 and class ≤ 3 , so $D_5(Q/N) = 1$: by Proposition 5.1 for class 3, and since $D_5 \subseteq D_3 = \gamma_3 = 1$ for class ≤ 2 . By Lemma 5.2, Q is then isomorphic to an immediate descendant of step size 1 of a group R of order 2^9 , of class 2 or 3 and 2-class ≥ 3 , with $\gamma_3(R)$ cyclic, and Q has class 3, cyclic centre, and its involution n of the centre lies in $\gamma_3(Q) \cap M_5(Q)$. **[computer]** The SmallGroups library contains 1 422 862 such groups R . For each, ANUPQ (GAP 4.15.1, ANUPQ 3.3.2) lists the immediate descendants of step size 1 up to isomorphism, 1 819 675 groups in all; 88 809 of them have class 3, and exactly 3400 of these have cyclic centre with involution in $\gamma_3 \cap M_5$ (by rank $d(Q) = 2, 3, 4, 5, 6$: 15, 301, 2756, 319, 9; none of rank 7). For each of the 3400 the lattice computation of §5 gives $D_4(Q) = D_5(Q) = 1$, in fact $D_n(Q) = \gamma_n(Q)$ for all n . This contradicts $D_5(Q) \neq 1$. $\square$

Checks of the order- 2^{10} search. The same pipeline at orders $2^6, \dots, 2^9$ (parents from SmallGroups of half the order) was compared with a direct pass over SmallGroups of that order: the numbers of class-3 groups with $|P_c| = 2$ and 2-class ≥ 4 (3, 27, 320, 4185), of those with cyclic centre, and of the candidates (3, 11, 64, 308) agree, also rank by rank. A second lattice method (the truncated Magnus algebra of the open questions, run on a pc-presentation, so independent of the multiplication table) gives the same answer on 3391 of the 3400 candidates. The search took 62.9 CPU-hours in GAP and the lattices 21.3 CPU-hours. Code, the proof of the lemma, the per-chunk and per-candidate tables: `compute/pq1024` (README.md, section “Result: order 2^{10} is closed”, and `results/`).

- Theorem 5.4 ([computer + paper]).** (1) $D_5(G) \subseteq \gamma_4(G)$ for every finite group G whose Sylow 2-subgroup has order at most 2^{10} ; in particular for every group of order less than 2048.
- (2) $D_n(G) = \gamma_n(G)$ for all n and every finite group G of order less than 512.
- (3) $D_5(G)/\gamma_5(G)$ has no 3-torsion for every finite group G whose Sylow 3-subgroup has order at most 3^7 ; in fact $D_5(Q) = \gamma_5(Q)$ for every 3-group Q of order at most 3^7 . (Extended to 3^9 in Theorem 6.12.)

Proof. (1) is Remark 2.3 with Propositions 5.1 and 5.3: every metabelian 2-group of class 3 and order $\leq 2^{10}$ is covered (orders $\leq 2^8$ by the lattice for all groups, 2^9 by the sieve and the lattice, 2^{10} by Lemma 5.2, ANUPQ and the lattice). For (2) and (3): a finite nilpotent group is the direct product of its Sylow subgroups and D_n, γ_n respect direct products, and for non-nilpotent finite G one reduces to the nilpotent quotients $G/\gamma_m(G)$ [paper; standard, see [MP09]]; groups of order < 512 have 2-part $\leq 2^8$, 3-part $\leq 3^5$ and p -part $\leq p^3$ for $p \geq 5$, where class ≤ 2 makes everything trivial. For the 3-part in (3): if Q is a 3-group of order $\leq 3^7$, then $Q/\gamma_5(Q)$ has class ≤ 4 and order $\leq 3^7$, hence is metabelian (class ≤ 3 always is, Remark 2.2; class 4 by Proposition 5.1), and Gupta–Tahara [GT85, Thm. 3.2(c)] ($D_{p+2} = \gamma_{p+2}$ for finitely generated metabelian p -groups, p odd) gives $D_5(Q/\gamma_5) = 1$, i.e. $D_5(Q) \subseteq \gamma_5(Q)$. This replaces the lattice computation for the 1202 candidates of order 3^7 (of which item (3) of Proposition 5.1 had checked a sample). The reduction for non-nilpotent groups in (2)–(3) is the standard one and has not been written out here. $\square$

Observations. In every group of class c computed, $D_{c+1} = 1$ and $D_k = \gamma_k$ at each step. For metabelian groups of class 3 the Jennings sieve removes almost everything (orders 64, 128, 256, 512: 111/114, 1116/1137, 21 105/21 325, 1 584 597/1 587 389), and the survivors have $D_4 = 1$ already. For 3-groups of class 4 the sieve removes little (9/71 at order 729, 107/1309 at order 2187). Computing time: about 3.6 hours wall clock, about 30–40 CPU-hours.

Remark 5.5 (pruning the generation tree; MATH-40). Let Q be a finite 2-group of class 3 with cyclic centre, $d = d(Q)$, $C = C_Q(\gamma_2(Q))$, $X = \gamma_2(Q)/(\gamma_2(Q) \cap Z(Q))$. The commutator map $X \times Q/C \rightarrow \gamma_3(Q)$ is a non-degenerate bilinear pairing into the cyclic group $\gamma_3(Q) \leq Z(Q)$; hence $X \cong Q/C$, a quotient of Q_{ab} , and X has rank $\leq \min(d, d(d-1)/2)$. If $|\gamma_3(Q)| = 2^e$, then Q_{ab} has two invariants $\geq 2^e$ (the map $(u, v) \mapsto [u, v, z]$ factors through $\Lambda^2 Q_{ab}$) and X has an element of order $\geq 2^e$. For a minimal counterexample Q (Lemma 5.2) with central involution n this gives conditions on every quotient $V = Q/N$ with $N \leq \Phi(Q)$ (with $n \in N \leq \Phi(Q)$ for the second), which pass to further quotients and so may be used to cut whole subtrees of the p -group generation tree: (i) there is a cyclic K , $\gamma_3(V) \leq K \leq \gamma_2(V) \cap Z(V)$, with $\text{rk}(\gamma_2(V)/K) \leq \min(d, d(d-1)/2)$; (ii) for such a K , $|Q| \geq 2|K| |A \vee B \vee E_1| |A \vee V_{ab} \vee E_2|$, where $A = \gamma_2(V)/K$, $B = V/C_V(\gamma_2(V))$, $E_1 = \mathbb{Z}/2^{e_V+1}$,

$E_2 = E_1^2$, $2^{e_V} = |\gamma_3(V)|$, and $\vee$ is the componentwise maximum of abelian invariants. In particular every such V of class 3 has $d + 4e_V + 2 \leq \log_2 |Q|$. **[paper]** (proofs in `compute/pq2048/README.md`). Condition (i) is vacuous for $d \leq 3$ and below order 2^{2d+1} ; condition (ii) is the one that cuts.

Next. At order 2^{11} the same argument needs as parents the groups of order 2^{10} of class ≤ 3 with cyclic γ_3 ; they are not in SmallGroups, and a rough extrapolation puts their number at $0.5\text{--}1.1 \cdot 10^9$. Remark 5.5 cuts whole subtrees without losing a candidate (checked: the same 3, 11, 64, 308 candidates at orders $2^6\text{--}2^9$) and leaves, by a sample, $10^5\text{--}2 \cdot 10^6$ parents; generating them is still about 100 CPU-days, mostly rank 5 **[computer; estimate]**. Ranks 2, 3 and 4 have been run through to order 2^{11} : 17, 586 and 52 568 candidates, all with $D_4 = D_5 = 1$ **[computer]**. For rank 4 the walk passes through 2 756 candidates at order 2^{10} , exactly the rank-4 count of Proposition 5.3; it kept 52 181 parents of order 2^{10} (and cut 9 937, 136 127, 946 663 vertices at orders 2^8 , 2^9 , 2^{10}) and took about 15 CPU-hours for the tree and 4.6 for the lattices (code: `compute/pq2048, run_rank.sh`). The pruned search used there was checked at order 2^{10} , where it reproduces the 3400 candidates of Proposition 5.3 exactly, rank by rank, while keeping 4.2% of the parents. Ranks ≥ 8 need no search (Proposition 5.7 below). So at order 2^{11} ranks 5–7 remain open **[open]**. For rank 7 the walk stops at the step-2 descendants of $(\mathbb{Z}/2)^7$, where the stock ANUPQ fails with an integer overflow, as at rank 8; rank 6 is next; rank 5, the bulk of the cost, is planned as a volunteer-computing (BOINC) run.

Order 2^{11} , rank ≥ 8 . Call a 2-group Q a *candidate* if it has class 3, cyclic centre with involution $n \in \gamma_3(Q) \cap M_5(Q)$, and 2-class $c \geq 4$ with $P_c(Q) = \langle n \rangle$. By Lemma 5.2 and Proposition 5.3, a 2-group of order 2^{11} and class 3 with $D_5 \neq 1$ is a candidate (its quotients by central subgroups of order 2 have order 2^{10} and class ≤ 3).

Lemma 5.6. *There is no candidate Q of order 2^{11} with $Q/P_3(Q) \cong \mathbb{Z}/4 \times (\mathbb{Z}/2)^7$.*

Proof [paper]. Such a Q has rank 8, so $|P_2(Q)| = |\Phi(Q)| = 8$; the series $P_2 \supsetneq P_3 \supsetneq P_4 \supseteq P_c \neq 1$ decreases strictly until it reaches 1, hence $|P_3(Q)| = 4$ and $P_4(Q) = \langle n \rangle$. (1) Q/P_3 is abelian, so $\gamma_2(Q) \leq P_3(Q)$. (2) Q has class 3, so $\gamma_3 = [\gamma_2, Q] \neq 1$, and $\gamma_3 \neq \gamma_2$ since Q is nilpotent and $\gamma_2 \neq 1$. Hence $|\gamma_2| = 4$, $\gamma_2 = P_3$, $|\gamma_3| = 2$, and $\gamma_3 = \langle n \rangle$, the only central subgroup of order 2. (3) $M_5(Q) = \prod_{i \geq 5} \gamma_i^{2^j} = Q^8 \gamma_2^4 \gamma_3^2 = Q^8$, since $\gamma_4 = 1$, $|\gamma_2| = 4$ and $|\gamma_3| = 2$. (4) Q/P_3 has exponent 4, so $Q^4 \leq P_3$ and $Q^8 \leq P_3^2 \leq P_4 = \langle n \rangle$. (5) As $n \in M_5(Q) = Q^8$, there is $x \in Q$ with $x^8 = n$. Then $x^4 \in P_3 = \gamma_2$ has order 4, so $\gamma_2 = \langle x^4 \rangle$. (6) For $g \in Q$, $[x, g] \in \gamma_2$, γ_2 is abelian and γ_3 is central, so $[x, g]^{x^i} = [x, g][x, g, x]^i$ and

$$[x^k, g] = \prod_{i < k} [x, g]^{x^i} = [x, g]^k [x, g, x]^{k(k-1)/2}.$$

For $k = 4$ this is $[x, g]^4[x, g, x]^6 = 1$, because γ_2 has exponent dividing 4 and $|\gamma_3| = 2$. So x^4 is central. (7) Then $\gamma_2 = \langle x^4 \rangle \leq Z(Q)$ and $\gamma_3 = [\gamma_2, Q] = 1$, contradicting class 3. $\square$

Proposition 5.7 ([computer + paper]). $D_5(Q) = 1$ for every 2-group Q of order 2^{11} , class 3 and rank $d(Q) \geq 8$.

Proof. Suppose not; then Q is a candidate, and $|\Phi(Q)| = 2^{11-d}$. Since $c \geq 4$, the strictly decreasing series $P_2 \supsetneq P_3 \supsetneq P_4 \neq 1$ gives $|\Phi(Q)| \geq 8$, so $d \leq 8$: rank ≥ 9 is impossible [paper]. Let $d = 8$. Then, as in the proof of Lemma 5.6, $c = 4$, $|P_3(Q)| = 4$ and $P_4(Q) = \langle n \rangle$; so $V = Q/P_3(Q)$ has order 2^9 , rank 8 and 2-class 2, and $R = Q/\langle n \rangle$ is an immediate descendant of step 1 of V . Moreover R has class 2: if it had class 3, Remark 5.5(ii) applied to R would give $d + 4e_R + 2 \leq 11$ with $e_R \geq 1$, i.e. $d \leq 5$ [paper]. [computer] The groups of order 2^9 and rank 8 are the 12 groups $\text{SmallGroup}(512, i)$, $10\,494\,201 \leq i \leq 10\,494\,212$ (a group V of rank n with $|\Phi(V)| = 2$ is determined by its squaring map $q: V/\Phi(V) \rightarrow \Phi(V)$, a nonzero quadratic form on $\mathbb{F}_2^n$ whose polar form is the commutator form, so such groups correspond to nonzero quadratic forms up to equivalence; if the polar form has rank $2k$ and radical R , then $q|_R = 0$ gives two classes for each $1 \leq k \leq n/2$ (Arf invariant 0 or 1) and $q|_R \neq 0$ one class for each $0 \leq k < n/2$, so $2\lfloor n/2 \rfloor + \lceil n/2 \rceil$ classes, 12 for $n = 8$; the same formula gives the numbers of groups of order 2^{n+1} and rank n in SmallGroups for $n = 2, \dots, 7$, namely 3, 4, 6, 7, 9, 10). Of the 11 non-abelian ones, 10 have trivial nucleus and hence no immediate descendants at all [OBr90]; for $\text{SmallGroup}(512, 10\,494\,202)$ the nucleus has dimension 1, but every immediate descendant of class ≤ 3 has class 3, so it has no descendant R of class 2. Both are read off the p -covering group, without computing automorphism groups; the two rules and their proofs are in `compute/pq2048/README.md` (section “Vertices without descendants”, Rules 0 and 2) [paper], checked there against full ANUPQ expansion. The remaining group $\text{SmallGroup}(512, 10\,494\,201) \cong \mathbb{Z}/4 \times (\mathbb{Z}/2)^7$ is excluded by Lemma 5.6. $\square$

An independent check of rank 8 by direct generation agrees: on $\mathbb{Z}/4 \times (\mathbb{Z}/2)^7$ the stock ANUPQ is estimated to need about 20 GB (a run was killed at 7 GB), while a version with a memory fix completes it in 2.1 GB, finding 8 immediate descendants of order 2^{10} ; the walk from all 12 groups then gives no candidate of order 2^{11} [computer].

Central products. A search beyond order 2^{11} may skip candidates that split as central products with a small factor. Let E be a finite 2-group of class 2 with $E/Z(E) \cong (\mathbb{Z}/2)^2$ and cyclic centre. Replacing E by the subgroup generated by two lifts a, b of a basis of $E/Z(E)$, which does not change the central product below, one has $E \in \{D_8, Q_8\}$ or $E \cong M_{2^{k+2}} = \langle a, b \mid a^{2^{k+1}} = b^2 = 1, a^b = a^{1+2^k} \rangle$, $k \geq 2$, with $Z(E) = \langle a^2 \rangle$ [paper]. For an embedding $\varphi: Z(E) \rightarrow Z(H)$ let $H \circ E = (H \times E)/\{(\varphi(z), z^{-1})\}$, $t = \varphi([a, b])$ and, for $E = M_{2^{k+2}}$, $s = \varphi(a^2)$.

Proposition 5.8 ([Lean]). *Let H be a group with $t \in \gamma_2(H)$, and $n \geq 2$.*

- (1) *If $E \in \{D_8, Q_8\}$, then $D_n(H \circ E) = D_n(H)$.*
- (2) *If $E = M_{2^{k+2}}$ and for some $g \in H$ the element $c = g^{-2}s$ satisfies $c^{2^j} - 1 \in \Delta(H)^{j+2}$ for all $j \geq 0$ (for instance $c^{2^j} \in \gamma_{j+2}(H)$), then $D_n(H \circ E) = D_n(H)$. This holds in particular if $s = g^2$ for some $g \in H$, central or not ($c = 1$), and if $s \in \gamma_2(H)$, $s^2 \in \gamma_3(H)$ ($g = 1$, $k = 2$).*

Proof. Let $Q \geq H$ be one of the groups below, generated by H and elements commuting with H , and $\gamma_2(Q) = \gamma_2(H)$. Then $D_n(Q) \subseteq \gamma_2(Q) \subseteq H$ and $D_n(H) \subseteq D_n(Q)$, so $D_n(Q) = D_n(H)$ follows once $\mathbb{Z}[H] \cap \Delta(Q)^n \subseteq \Delta(H)^n$. For that it suffices to have a left $\mathbb{Z}[H]$ -linear $q: \mathbb{Z}[Q] \rightarrow \mathbb{Z}[H]$, the identity on $\mathbb{Z}[H]$, with $q(\Delta(Q)^n) \subseteq \Delta(H)^n$. Here $\gamma_2(H \circ E) = \gamma_2(H)\langle t \rangle = \gamma_2(H)$.

(1) Let $E = D_8$, $c = [a, b]$, $\tau = c - 1$ (so $\tau^2 = -2\tau$), and $N = \mathbb{Z}[E/\langle b \rangle]$, free over $\mathbb{Z}[\langle c \rangle]$ on e_0 and $f = ae_0 - e_0$. Then $\mathbb{Z}[H] \otimes_{\mathbb{Z}[\langle c \rangle]} N$ is a $\mathbb{Z}[Q]$ -module, and $q(u) = (\text{id} \otimes p)(u(1 \otimes e_0))$ with $p(xe_0 + yf) = x$ is as required. Indeed, $\Delta(Q)^n$ is spanned by products of i factors from $\Delta(H)$ and $n - i$ from $\Delta(E)$, and on N in the coordinates (x, y) one has $a - 1: (x, y) \mapsto (0, x - 2y)$ and $b - 1: (x, y) \mapsto (\tau y, \tau y)$. By induction, $p(\Delta(E)^j N) = \tau^{\lceil j/2 \rceil} \mathbb{Z}[\langle c \rangle]$ for $j \geq 1$; as $t - 1 \in \Delta(H)^2$, $q(\Delta(Q)^n) \subseteq \sum_i \Delta(H)^i (t - 1)^{\lceil (n-i)/2 \rceil} \subseteq \Delta(H)^n$. For Q_8 : $u = ab \in D_8$ has $u^2 = t$, so $H' = H\langle u \rangle \leq H \circ D_8$ and $D_n(H') = D_n(H)$. If a', b' are the generators of the factor D_8 of $H' \circ D_8$, the elements $a'u, b'u$ generate a Q_8 meeting H' in $\langle t \rangle$, so $H \circ Q_8 \leq H' \circ D_8$ and $D_n(H \circ Q_8) \subseteq D_n(H') \cap H = D_n(H)$, using (1) for H' .

(2) Let $H'' = H\langle r \rangle$ with r central, $r^2 = s$, $\langle r \rangle \cap H = \langle s \rangle$. With a', b' the generators of the factor D_8 of $H'' \circ D_8$, the elements $a'r, b'$ generate a copy of E meeting H'' in $\langle s \rangle$, and $H\langle a \rangle \cong H''$ inside $H \circ E$; by (1), $D_n(H \circ E) = D_n(H'')$. Now $\mathbb{Z}[H''] = \mathbb{Z}[H] \oplus \mathbb{Z}[H]r$; put $q(u_0 + u_1 r) = u_0 + u_1 g$. As r is central, $\Delta(H'')^n = \sum_i \Delta(H)^{n-i} (r - 1)^i \mathbb{Z}[H'']$, so it suffices that $f_i = q((r - 1)^i) \in \Delta(H)^i$ for all i . In the abelian group $\langle g, s, r \rangle$ put $y = rg^{-1}$, so $y^2 = c$, and $x = g - 1$, $\eta = y - 1$, $\tau = c - 1$. Then $r - 1 = x + \eta + x\eta$ and $f_i = \sum_{\alpha+\beta \geq i} m_{\alpha\beta} x^\alpha e_\beta$ with $e_b = \pi(\eta^b)$, $\pi(v_0 + v_1 y) = v_0 + v_1$. From $\eta^2 = \tau - 2\eta$ we get $e_1 = 0$, $e_2 = \tau$ and $e_{b+2} = \tau e_b - 2e_{b+1}$. The identity $2(c^{2^j} - 1) = (c^{2^{j+1}} - 1) - (c^{2^j} - 1)^2$ and the hypothesis give $2^m \tau \in \Delta(H)^{m+2}$ for all m . Hence, by induction on $b \geq 1$, $2^m e_b \in \Delta(H)^{b+m}$ for all m ; in particular $e_b \in \Delta(H)^b$. $\square$

The condition $c \in \gamma_2(H)$, i.e. $s \in \Phi(H)$ for a 2-group, cannot be dropped from the method: if $s \notin \Phi(H)$, no such q for H'' satisfies $q(\Delta(H'')^2) \subseteq \Delta(H)^2$ [paper]. [computer] For the 386 candidates H of orders 2^6 – 2^9 , $t = n \in \gamma_3(H)$. The 291 with $|Z(H)| \geq 8$ have s a square in the cyclic $Z(H)$. Of the 95 with $Z(H) \cong \mathbb{Z}/4$, 78 have $s \in \gamma_2(H)$ and $s^2 \in \gamma_3(H)$, and the other 17 have $s = g^2$ with $g \in H$ not central. So by Proposition 5.8, $D_n(H \circ E) = D_n(H)$ for all of them, all n and $E \in \{D_8, Q_8, M_{16}\}$. Independently, the lattice computation gives $D_4 = D_5 = 1$ for all 1 544 such products.

Gluing along a proper subgroup. For $C \leq Z(E)$ and an embedding $\varphi: C \rightarrow Z(H)$ let $H \circ_C E = (H \times E)/\{(\varphi(z), z^{-1}) : z \in C\}$. Then $H' = HZ(E)$ centralises E , $H \circ_C E = H' \circ E$ is a gluing along all of $Z(E)$ (with φ the inclusion, so $s = a^2 \in H'$ for $E = M_{2^{k+2}}$), and $\gamma_i(H') = \gamma_i(H)$ for $i \geq 2$. For $E = M_{2^{k+2}}$, $H' \circ E$ can be written as $H \circ_C E$ with $C \neq Z(E)$ exactly when $s \notin \Phi(H')$, which is where the method of part (2) for H' fails by the remark above. The projection then has to go to H instead of H' .

Lemma 5.9 ([Lean]). *Let $K = H\langle r \rangle$ with r central, $|K : H| = m$, and $r^m = \sigma = g^m$ for some $g \in H$, not necessarily central. Then $\mathbb{Z}[H] \cap \Delta(K)^n = \Delta(H)^n$ for all n , and $D_n(K) = D_n(H)$ for $n \geq 2$.*

Proof. $\mathbb{Z}[K] = \bigoplus_{0 \leq j < m} \mathbb{Z}[H]r^j$; put $q(\sum_j u_j r^j) = \sum_j u_j g^j$. As $r^m = g^m \in H$, $q(ur^j) = ug^j$ for all $j \in \mathbb{Z}$, so $q(up(r)) = up(g)$ for every Laurent polynomial p . As r is central, $\Delta(K)^n$ is spanned by the $x(r-1)^i r^j$ with $x \in \Delta(H)^{n-i}$, and $q(x(r-1)^i r^j) = x(g-1)^i g^j \in \Delta(H)^n$. Hence $\mathbb{Z}[H] \cap \Delta(K)^n = q(\mathbb{Z}[H] \cap \Delta(K)^n) \subseteq \Delta(H)^n$; the reverse inclusion is clear, and $D_n(K) \subseteq \gamma_2(K) = \gamma_2(H) \subseteq H$ gives the rest. $\square$

For $m = 2$ this is the case $c = 1$ of Proposition 5.8(2). Let $E = M_{2^{k+2}}$, $C = \langle a^{2^{j+1}} \rangle$ with $1 \leq j < k$, and $\sigma = \varphi(a^{2^{j+1}})$. If $t \in \gamma_2(H)$ and $\sigma = g^{2^{j+1}}$ for some $g \in H$, then $D_n(H \circ_C E) = D_n(H)$ for $n \geq 2$ **[paper]**: in the proof of (2) for H' one has $H'' = H'\langle r \rangle = H\langle r \rangle$ with $r^{2^{j+1}} = \sigma$ and $\langle r \rangle \cap H = \langle \sigma \rangle$, so $|H'' : H| = 2^{j+1}$, and Lemma 5.9 applies with $m = 2^{j+1}$. For instance, $H \circ_{\langle t \rangle} M_{16}$ needs $t = g^4$. The hypothesis $t \in \gamma_2(H)$ costs nothing for the search: if $H \circ_C E$ is a candidate, t is the involution of its cyclic centre, so $t = n \in \gamma_3(H \circ_C E) = \gamma_3(H)$. **[computer]** Of the 386 candidates of orders 2^6 – 2^9 , 20 split as central products (5 of order 2^8 , 15 of order 2^9), with 212 factors E in all. The gluing is proper in 16 of these splittings, belonging to 4 candidates, always with M_{32} or M_{64} ; Lemma 5.9 covers 4 of them. Each of the 20 candidates also has a splitting covered by Proposition 5.8, so at these orders a search that skips split candidates does not need proper gluing (code: `compute/central`, `decomp.g`; wiki, MATH-186). The other 12 splittings would need an analogue of (2) for $m = 4$ (taking H maximal in H' , i.e. $j = 1$) with $g^{-4}\sigma \neq 1$; it cannot go through a projection to H in general, since for $Q = D_8 \circ_{\langle t \rangle} M_{16}$ one has $\mathbb{Z}[D_8] \cap \Delta(Q)^n \neq \Delta(D_8)^n$ at $n = 4, 5$ (index 4), although $D_n(Q) = D_n(D_8)$ **[computer] [open]**.

3-torsion. For 3-torsion the computation is no longer needed up to order 3^7 (Theorem 5.4(3)); orders 3^8 and 3^9 and an infinite family are treated in §6. Quotients of the known examples (Rips, $D_4 \neq \gamma_4$ at order 2^{38} ; Bartholdi–Mikhailov at order 3^{494}) could calibrate how far a counterexample to Question 1.1 must be.

6. 3-TORSION: ORDERS 3^8 AND 3^9 , AND GROUPS WITH G/γ_2 OF EXPONENT 3

Theorem 5.4(3) settles 3-torsion up to order 3^7 by metabelianity. Here we go further: we prove $D_5 = \gamma_5$ for all 3-groups of order $\leq 3^9$ (Theorem 6.12) and, on paper, for every finite 3-group Q with $Q/\gamma_2(Q)$ of exponent 3, of any order (Theorem 6.6). Details, the computations and two independent line-by-line reviews are in our wiki (MATH-120, MATH-127); code in `compute/magnus`. Throughout, the *type* of a group Q of class c is $(a_1, \dots, a_c)$ with $p^{a_i} = |\gamma_i(Q)/\gamma_{i+1}(Q)|$, and $L(Q) = \bigoplus_i \gamma_i/\gamma_{i+1}$ is its Lie ring.

Lemma 6.1. *Let p be odd and Q a finite p -group of class $\leq p+1$. Then $D_{p+2}(Q) \subseteq Q'' \cap M_{p+2}(Q)$. In particular, for $p=3$ and class ≤ 4 , $D_5(Q) \subseteq Q'' \cap M_5(Q)$, and $D_5(Q) \neq 1$ needs $Q'' \neq 1$, hence $d(Q) \geq 3$.*

Proof [paper]. Q/Q'' is metabelian of class $\leq p+1$, so $D_{p+2}(Q/Q'') = \gamma_{p+2}(Q/Q'') = 1$ by [GT85, Thm. 3.2], and $D_{p+2}(Q) \subseteq Q''$ by functoriality. $D_n \subseteq M_n$ because $\mathbb{Z}[Q] \rightarrow \mathbb{F}_p[Q]$ maps Δ^n into $\Delta_{\mathbb{F}_p}^n$. If $d(Q) \leq 2$ then γ_2/γ_3 is cyclic and $Q'' \subseteq \gamma_5 = 1$ (see Lemma 6.3). $\square$

This replaces the appeal to [MP09, Thm. 2.28] in the paragraph on 3-torsion of §5.

Lemma 6.2 (minimal counterexample, $p=3$). *Let Q be a 3-group of least order with $D_5(Q) \neq \gamma_5(Q)$. Then Q has class 4, $Q'' \neq 1$ and $d(Q) \geq 3$; $Z(Q)$ is cyclic and $D_5(Q) = \Omega_1(Z(Q)) = \langle z \rangle$ has order 3, with $z \in Q'' \cap M_5(Q)$; the last nontrivial term $P_c(Q)$ of the lower exponent-3 central series equals $\langle z \rangle$, so Q is an immediate descendant of step size 1 of $Q/\langle z \rangle$.*

Proof [paper]. If Q had class ≥ 5 , then $Q/\gamma_5(Q)$ would be a smaller counterexample; for class ≤ 3 , $D_5 \subseteq D_4 = \gamma_4 = 1$ (Passi: $D_4 = \gamma_4$ without 2-torsion [Pas68]). So Q has class 4, and Lemma 6.1 gives $1 \neq D_5(Q) \subseteq Q'' \cap M_5(Q)$. The rest is the argument of Lemma 5.2: $D_5(Q)$ lies in every normal N with $D_5(Q/N) = 1$, in particular in every central subgroup of order 3. $\square$

A Lie-ring lemma.

Lemma 6.3. *Let p be odd and G a group with $|\gamma_3(G)/\gamma_4(G)| = p$ and $\gamma_2(G)/\gamma_3(G)$ generated by at most two elements. Then $[\gamma_2(G), \gamma_2(G)] \subseteq \gamma_5(G)$. In particular, G is metabelian if it has class ≤ 4 .*

Proof [paper; Lean] (in Lean the same computation is done in the group $G/\gamma_5(G)$, §4). The image of $[\gamma_2, \gamma_2]$ in γ_4/γ_5 is spanned by the brackets $[L_2, L_2] \subseteq L_4$ of $L = L(G)$, so it suffices to show $[L_2, L_2] = 0$. If L_2 is cyclic, the alternating bracket $L_2 \times L_2 \rightarrow L_4$ vanishes. Otherwise $pL_3 = 0$, hence $pL_4 = 0$ ($L_4 = [L_3, L_1]$), and dividing L by the ideal $pL_1 \oplus pL_2 \oplus 0 \oplus H \oplus L_{\geq 5}$, where $H \subset L_4$ is a hyperplane missing a given nonzero bracket $[u, v]$, we

get a graded $\mathbb{F}_p$ -Lie algebra $M = M_1 \oplus \cdots \oplus M_4$ generated by M_1 with $\dim M_2 = 2$, $\dim M_3 = \dim M_4 = 1$ and $[M_2, M_2] \neq 0$. We show that this is impossible.

Write $[z, z'] = \omega(z, z')w$ for $z, z' \in M_2$, $M_4 = \langle w \rangle$, $M_3 = \langle t \rangle$, $[x, t] = \varphi(x)w$, $[x, z] = \beta(x, z)t$ and $[x, y] = \alpha(x, y)$ for $x, y \in M_1$, $z \in M_2$. A nonzero alternating form on a plane is nondegenerate, so ω is; $\varphi \neq 0$ since $M_4 = [M_1, M_3]$. Pick e with $\varphi(e) = 1$ and let $K = \ker \varphi$. The Jacobi identity for $x, y \in M_1$, $z \in M_2$ reads

$$\omega(\alpha(x, y), z) = \varphi(x)\beta(y, z) - \varphi(y)\beta(x, z).$$

For $x, y \in K$ the right side vanishes, so $\alpha(K, K) = 0$ and $M_2 = \alpha(e, K)$; for $x = e$, $y = k \in K$ it gives $\beta(k, z) = \omega(\alpha(e, k), z)$. The Jacobi identity for $e, k_1, k_2 \in K$, with $u_i = \alpha(e, k_i)$ and $\alpha(k_1, k_2) = 0$, gives $\beta(k_1, u_2) - \beta(k_2, u_1) = 0$, i.e. $2\omega(u_1, u_2) = 0$. As $p \neq 2$ and the u_i run over M_2 , $\omega = 0$, a contradiction. $\square$

An independent proof for $|\gamma_2/\gamma_3| = p^2$ (both forms ω_1, ω_2 of $[L_1, L_1] \rightarrow L_2$ and an exterior-algebra computation) is in the wiki (MATH-127, part 3).

Remark 6.4. (1) The hypothesis $p \neq 2$ is needed: SmallGroup(128, i), $i = 928, \dots, 937$, have type $(3, 2, 1, 1)$ and $|G''| = 2$, and there are 108 such groups of order 2^8 **[computer]**.
 (2) The other two hypotheses are sharp at order 3^8 : of the 904 non-metabelian 3-groups of class 4 and order 3^8 , 15 have type $(3, 3, 1, 1)$ and 889 type $(3, 2, 2, 1)$ **[computer]**; an explicit Lie algebra of type $(3, 3, 1, 1)$ with $[M_2, M_2] \neq 0$ is in the wiki **[paper]**.
 (3) Check: for all groups of orders 3^5 – 3^8 of class ≥ 4 the hypothesis holds in 13 077 groups and the conclusion in all of them; a referee's scan of all 1 396 077 groups of order 3^8 and all groups of order 7^7 found 24 215 and 4 408 groups with $|\gamma_2/\gamma_3| = p^2$, $|\gamma_3/\gamma_4| = p$, and no violation **[computer]**.

Proposition 6.5 (**[Lean]**; the examples for $p = 3$: **[computer]**). *Let p be odd and Q a p -group of class 4.*

- (1) *If $|Q| \leq p^7$, or $|Q| = p^8$ and $d(Q) \geq 4$, then Q is metabelian.*
- (2) *If $|Q| = p^9$ and $Q'' \neq 1$, then $d(Q) \in \{3, 4\}$; for $d(Q) = 4$ the type is $(4, 3, 1, 1)$ or $(4, 2, 2, 1)$, and for $p = 3$ both occur ($Q_0 \times C_3$ with Q_0 of order 3^8 as in Remark 6.4(2)).*

Proof. Let (a_1, a_2, a_3, a_4) be the type. By Lemma 6.3, $Q'' \neq 1$ needs $a_3 \geq 2$ or γ_2/γ_3 with at least three generators; as γ_2/γ_3 is not cyclic, $a_2 \geq 2$, so $a_2 \geq 3$ or $a_3 \geq 2$. Moreover $a_1 \geq d(Q)$, $a_4 \geq 1$, and for $d(Q) = 2$ the group γ_2/γ_3 is cyclic. So $d(Q) \geq 3$ and $|Q| \geq p^{d(Q)+5}$, which gives (1) and the first part of (2); at p^9 and $d = 4$ the bound is attained exactly by the two types listed. At p^9 and $d = 5$ the only type with $a_2 \geq 2$ is $(5, 2, 1, 1)$, and such groups are metabelian. $\square$

In particular the computer statement “every 3-group of class 4 and order $3^5, 3^6, 3^7$ is metabelian” of Proposition 5.1 now has a proof, and so does the observation that all non-metabelian groups of class 4 and order 3^8 have rank 3.

Groups with G/γ_2 of exponent 3.

Theorem 6.6 ([paper; Lean modulo Tahara]; relies on [Tah81, Thm. 6.1]).

Let Q be a finite 3-group with $Q/\gamma_2(Q)$ of exponent 3. Then $D_5(Q) = \gamma_5(Q)$. More generally, $D_5(G) = \gamma_5(G)$ for every group G with $G/\gamma_2(G)$ finite of exponent 3.

This covers non-metabelian groups of class 4 of every rank and order; for instance all 904 non-metabelian groups of class 4 and order 3^8 have $Q/\gamma_2 \cong C_3^3$ [computer].

Proof. Reductions. For G as in the second statement, $\bar{G} = G/\gamma_5(G)$ is nilpotent with finite abelianization, hence finite; its lower central factors are images of tensor powers of $\bar{G}_{ab}$, so $\bar{G}$ is a 3-group, and $D_5(G) \subseteq \gamma_5(G)$ follows from $D_5(\bar{G}) = 1$ by functoriality. So let Q be a finite 3-group with $\gamma_5(Q) = 1$ and Q/γ_2 of exponent 3; we show $D_5(Q) = 1$. Tahara’s Theorem 6.1 is stated for any finite group with an N -series ending at $H_5 = 1$ [Tah81, §6, p. 159], so class ≤ 3 needs no separate treatment. We write γ_3 and γ_4 additively where convenient.

Identities. (i) $\gamma_2/\gamma_3, \gamma_3/\gamma_4, \gamma_4$ are elementary abelian: they are spanned by the values of bilinear maps with one argument in Q/γ_2 . (ii) $P: Q/\gamma_2 \rightarrow \gamma_2/\gamma_3, x \mapsto x^3\gamma_3$, and $\pi: \gamma_2/\gamma_3 \rightarrow \gamma_3/\gamma_4, a \mapsto a^3\gamma_4$, are homomorphisms. (iii) For $x \in Q, a \in \gamma_2$:

$$[x^3, a] = [x, a]^3 = [x, a^3], \quad [a, x^3] = [a, x]^3.$$

Indeed, for $c = [x, a] \in \gamma_3, c^x = c[c, x]$ with $[c, x]$ central, so $[x^3, a] = c^3[c, x]^3 = c^3$ as γ_4 has exponent 3; and $[x, a^3] = c c^a c^{a^2} = c^3$ as $[\gamma_3, \gamma_2] = 1$. Consequently, if $a \in \gamma_2$ and $a^3 \in \gamma_4$, then $[x^3, a] = [x, a^3] = 1$.

Tahara’s description. Choose bases of uniqueness: $x_1, \dots, x_s$ of Q/γ_2 such that $x_1, \dots, x_m$ span a complement of $\ker P$ and $x_{m+1}, \dots, x_s$ span $\ker P$ (so $x_i^3 \in \gamma_3$ for $i > m$); $x_{2k} = x_k^3$ for $k \leq m$, completed to a basis $x_{21}, \dots, x_{2t}$ of γ_2/γ_3 . All orders $d(i), e(k), f(l)$ equal 3, and the constants b_{ik} of $x_i^3 \equiv \prod_k x_{2k}^{b_{ik}} \pmod{\gamma_3}$ are $b_{ik} = \delta_{ik}$ for $i \leq m, b_{ik} = 0$ for $i > m$. By [Tah81, Thm. 6.1] (checked against the original, pp. 159–161), $D_5(Q)$ is generated by the elements

$$g = \sum_{i < j} u_{ij} [x_i^3, x_j] + \sum_{i, k} v_{ik} \sum_{l > k} b_{il} [x_{2l}, x_{2k}] + \sum_{i \leq j \leq k} w_{ijk} [x_i^3, x_j, x_k]$$

for integer families (u, v, v', w, w', w'') satisfying Tahara’s conditions (A)–(F) (Tahara writes g with minus signs; the conditions are homogeneous, so the subgroup is the same). With all orders 3 we use: (B) $\sum_{h > i} u_{ih} b_{hk} - \sum_{h < i} u_{hi} b_{hk} + 3v_{ik} + 3v'_{ik} = 0$; (C) $3 \mid u_{ij}$; (D) $3v_{ik} - \sum_{h \leq i} w_{hii} b_{hk} -$

$\sum_{h>i} w''_{iih} b_{hk} \equiv 0$ and, for $i < j$, $\sum_{h \leq i} w_{hij} b_{hk} + \sum_{i < h \leq j} w'_{ihj} b_{hk} + \sum_{h>j} w''_{ijh} b_{hk} \equiv 0 \pmod{3}$; (E) $\sum_k v'_{ik} d_{kl} \equiv$ a combination of u -terms and of terms $w_{gih} \alpha_l^{(gh)}$, $w_{ghi} \alpha_l^{(gh)}$, $w'_{igh} \alpha_l^{(gh)}$ (mod 3), where $x_{2k}^3 \equiv \prod_l x_{3l}^{d_{kl}}$ and $[x_g^3, x_h] \equiv \prod_l x_{3l}^{\alpha_l^{(gh)}}$ (mod γ_4); (F) $\sum_i v_{ik} b_{ik} \equiv 0$ and $\sum_i v_{ik} b_{il} + \sum_i v_{il} b_{ik} \equiv 0 \pmod{3}$ for $k < l$. Condition (C) is Tahara's (C) for $p = 3$ and $d(i) = d(j)$, where $\binom{3}{3} = 1$ is not divisible by 3; for $p \geq 5$ it is empty.

$g = 0$ for every admissible family. (1) Putting $b_{hk} = \delta_{hk}$ ($k \leq m$) into (D) shows that every coefficient w, w', w'' whose distinguished index (the first, middle, last one respectively) is $\leq m$ is divisible by 3. (2) Hence the w -part vanishes: for $i \leq m$ by (1) and since γ_4 has exponent 3; for $i > m$ because $x_i^3 \in \gamma_3$ and $[x_i^3, x_j, x_k] \in \gamma_5 = 1$. (3) Writing $u_{ij} = 3u'_{ij}$ by (C), identity (iii) gives $u_{ij}[x_i^3, x_j] = u'_{ij}[x_i^3, x_j^3]$, which vanishes unless $i, j \leq m$. Put $Y_k = x_k^3$ ($k \leq m$). (4) The v -part equals $\sum_{k < i \leq m} v_{ik}[Y_i, Y_k]$. (5) With U the skew matrix $U_{ik} = u'_{ik}$ ($i < k$), (B) gives $v'_{ik} = -v_{ik} - U_{ik}$ for $k \leq m$ and $v'_{ik} = -v_{ik}$ for $k > m$; (F) gives $v_{kk} \equiv 0$, $v_{lk} \equiv -v_{kl}$ for $k, l \leq m$ and $v_{kl} \equiv 0$ for $k \leq m < l$; in (E) all w -terms vanish modulo 3 by (1), since for a distinguished index $g > m$ we have $\alpha^{(gh)} \equiv 0$. So (E) says that $a_i = \prod_k x_{2k}^{v'_{ik}}$ satisfies $a_i^3 \in \gamma_4$. With $M = v + U$ (skew modulo 3 on indices $\leq m$) we get $a_i \gamma_3 = -\rho_i$, $\rho_i = \sum_{k \leq m} M_{ik} Y_k \in \ker \pi$. (6) Altogether

$$g = \sum_{k < i \leq m} M_{ik}[Y_i, Y_k] = \frac{1}{2} \sum_{i \leq m} [x_i^3, r_i]$$

with $r_i \in \gamma_2$ representing ρ_i (dividing by 2 is legitimate in γ_4 of exponent 3), and $[x_i^3, r_i] = [x_i, r_i^3] = 1$ by (iii), since $r_i^3 \in \gamma_4$ is central. So $g = 1$. $\square$

Remark 6.7. The exponent hypothesis is used in three places: in (C) (for $d(j) \geq 3d(i)$ Tahara's condition is empty and terms $u_{ij}[x_i^{d(i)}, x_j^3]$ with $x_j^3 \notin \gamma_2$ appear), in the choice of basis (Tahara's (D), (E) hold only modulo $(d(i), e(k)), (d(i), f(l))$), and in (iii). The next case, $Q/\gamma_2 \cong C_{3^a} \times C_3^{s-1}$, is settled by Theorem 6.8 below. Check of the theorem: 169 random non-metabelian groups of class 4 with $Q/\gamma_2 \cong C_3^3$ or C_3^4 , of orders 3^{10} – 3^{79} , 29 of them with $\Omega_1(Z) \cap Q'' \cap M_5 \neq 1$: identity (iii) holds and $D_5 = 1$ in all of them (truncated Magnus algebra modulo 3^{19} with the exactness certificate) [computer].

Theorem 6.8 ([paper; Lean modulo Tahara]; relies on [Tah81, Thm. 6.1]; checked by an independent line-by-line review). *Let Q be a finite 3-group with $Q/\gamma_2(Q) \cong C_{3^a} \times C_3^{s-1}$ for some $a \geq 1$, i.e. with at most one invariant of Q/γ_2 larger than 3. Then $D_5(Q) = \gamma_5(Q)$; the same holds for every group G with $G/\gamma_2(G)$ of this form.*

Sketch. For $a = 1$ this is Theorem 6.6; let $a \geq 2$, $\gamma_5(Q) = 1$, and let x_s have order 3^a modulo γ_2 . Since $\Lambda^2(C_{3^a} \times C_3^{s-1})$ has exponent 3, the factors γ_2/γ_3 , γ_3/γ_4 , γ_4 are still elementary abelian, so identities (i)–(iii) of the proof of

Theorem 6.6 hold for all $x \in Q$, and moreover $[x^{3^n}, y] = [x, y]^{3^n}$ for $n \geq 2$. The class $x_s^{3^a} \gamma_3$ does not depend on the choice of x_s , so there are two cases. If $x_s^{3^a} \notin \gamma_3$, Tahara's (B) forces $3 \mid u_{is}$, every term involving $x_s^{3^a}$ vanishes by the identity above, and the proof of Theorem 6.6 goes through. If $x_s^{3^a} \in \gamma_3$, the coefficients u_{is} are unrestricted. For $a \geq 3$ all new terms are 3^a -th powers of elements of γ_3 and vanish. For $a = 2$ the new term $\sum_i u_{is}[x_i^9, x_s]$ of g cancels only half of the pairing $\frac{1}{2} \sum_i [x_i^3, r_i]$, which (E) now gives as $-\sum_i u_{is}[x_i, x_s]^9$; the remainder is $\frac{1}{2}[r_s^3, x_s]$ with $r_s = \prod_i x_i^{3u_{is}}$, and condition (E) for the row s , which sees u_{is} through (B) with $d(s)/d(i) = 3$, gives $r_s^3 \in \gamma_4$. So $g = 1$ for every admissible family. (This step is not void: there are groups of order 3^{16} with $x_s^9 \in \gamma_3$ and $[x_i^9, x_s] \neq 1$ [computer].) Here $\alpha^{(ss)} = 0$ is read off Tahara's generators of types (V), (VI) [Tah81, pp. 150–151]. Details: wiki, MATH-135. $\square$

Consequently a minimal 3-group with $D_5 \neq \gamma_5$ has at least two invariants ≥ 9 in Q/γ_2 , so $|Q/\gamma_2| \geq 3^5$. Checks: random non-metabelian groups of class 4 with $Q/\gamma_2 \cong C_9 \times C_3^{s-1}$, $s = 2, 3, 4$, and $C_{27}, C_{81} \times C_3^{s-1}$, including the case $x_s^{3^a} \in \gamma_3$ [computer; runs in progress].

The p -covering group. For a p -group P of p -class c let P^* be its p -covering group, $M \leq Z(P^*)$ the multiplier (elementary abelian, $P^*/M \cong P$) and $N = P_{c+1}(P^*) \leq M$ the nucleus [OBr90]. The immediate descendants of step size 1 are the $Q = P^*/U$, $U < M$ of index p with $N \not\leq U$, and then $\langle z \rangle = M/U$ is central of order p with $Q/\langle z \rangle \cong P$.

Lemma 6.9 (a filter on parents). *If one of $M \cap P^{*''}$, $M \cap M_5(P^*)$, N and, when P has class 3, $\gamma_4(P^*)$, lies in $\gamma_5(P^*)$, then no immediate descendant Q of step size 1 of P is of class 4 with $z \in Q'' \cap M_5(Q)$.*

Proof [paper]. γ_n , derived subgroups and the Jennings subgroups $M_n = \prod_{i \leq n} \gamma_i^{p^i}$ pass to quotients: $X(P^*/U) = X(P^*)U/U$. Class 4 needs $\gamma_5(P^*) \leq U$ (and $\gamma_4(P^*) \not\leq U$ if P has class 3); $z \in Q''$ needs $M \cap P^{*''} \not\leq U$; $z \in M_5(Q)$ needs $M \cap M_5(P^*) \not\leq U$; and $N \not\leq U$. $\square$

Theorem 6.10 ([paper]). *Let p be odd and P a finite p -group of class 3 with $|\gamma_2(P)/\gamma_3(P)| = p^2$ and $|\gamma_3(P)| = p$. Then $P^{*''} = 1$. In particular every immediate descendant of step size 1 of P is metabelian, and P is rejected by Lemma 6.9.*

Proof. $P'' \subseteq \gamma_4(P) = 1$, so $P^{*''} \leq M$, and as the subgroups U of index p in M intersect trivially it suffices to show $Q'' = 1$ for $Q = P^*/U$. Here $\gamma_4(Q) \leq \langle z \rangle$. If $z \notin \gamma_2(Q)$, then $\gamma_2(Q) \cong \gamma_2(P)$ is abelian. If $z \in \gamma_2(Q) \setminus \gamma_4(Q)$, then $\gamma_4(Q) = 1$ and $Q'' \subseteq \gamma_4(Q) = 1$. If $z \in \gamma_4(Q)$, then Q has the same factors γ_2/γ_3 , γ_3/γ_4 as P , and Lemma 6.3 gives $Q'' \subseteq \gamma_5(Q) = 1$. (Lemma 6.3 does not apply to P^* itself, whose lower central factors are larger.) $\square$

Remark 6.11. (1) For $p = 2$ the theorem fails: SmallGroup(64, 138) has type (3, 2, 1), while its 2-covering group has order 2^{13} , class 4 and

- $|P^{*''}| = 2$; likewise $\text{SmallGroup}(128, i)$, $i = 522, 599, 621, 1755$, and 12 groups of order 2^8 **[computer]**.
- (2) Check: the hypothesis holds for 13 315 groups of order 3^8 and 2 754 of order 7^7 , and $P^{*''} = 1$ in all of them **[computer]**.
- (3) The theorem does not say that such P are terminal. Of the 7 824 groups of order 3^8 and type $(5, 2, 1)$ ($\text{SmallGroup}(3^8, i)$, $1\,387\,558 \leq i \leq 1\,395\,381$), 190 have nucleus of order 3, e.g. $\text{SmallGroup}(81, 9) \times 3_+^{1+2} \times C_3$, and all of their descendants are metabelian of type $(5, 2, 1, 1)$ **[computer]**. For P/γ_2 elementary abelian, nonterminality needs $|Z(P) \cap \gamma_2(P)| = p^2$, $\gamma_2(P)$ elementary abelian and $P^p \leq Z(P)$, and for $p = 3$ a further linear condition over $\mathbb{F}_3$ **[paper]**; these conditions hold for the 190 but also for 573 terminal groups, and an exact criterion is open **[open]**.

Orders 3^8 and 3^9 .

Theorem 6.12 ([computer + paper]). $D_5(Q) = \gamma_5(Q)$ for every 3-group Q of order at most 3^9 . Hence $D_5(G)/\gamma_5(G)$ has no 3-torsion for every finite group G whose Sylow 3-subgroup has order at most 3^9 .

Proof. By Theorem 5.4(3) a counterexample Q of least order has order 3^8 or 3^9 , and Lemma 6.2 applies to it; for order 3^9 we may use the result for order 3^8 . The deduction for G is as in Theorem 5.4.

Order 3^8 . Every group is in the SmallGroups library (package `sglppow`). Of the 1 396 077 groups, 904 are non-metabelian of class 4; all have rank 3 (as they must, Proposition 6.5), $|Z| = |Q''| = |\gamma_4| = 3$ and $Q/\gamma_2 \cong C_3^3$, types $(3, 2, 2, 1)$ (889) and $(3, 3, 1, 1)$ (15) **[computer]**. So Theorem 6.6 gives $D_5 = 1$ for all of them. Independently, 699 of them have $z \in Q'' \cap M_5$, and for each the truncated Magnus algebra (121 monomials, modulo 3^{19} , with the exactness certificate $3^{18}\Delta \subseteq \Delta^5 + 3^{19}$) gives $z \notin D_5$ **[computer]**.

Order 3^9 . Let Q be a counterexample and $P = Q/\langle z \rangle$, of order 3^8 and class 3 or 4. By Proposition 6.5(2), $d(Q) \in \{3, 4\}$; the case $d(Q) = 5$, type $(5, 2, 1, 1)$, is excluded on paper (alternatively, by Theorem 6.10: all 7 824 possible parents are rejected by Lemma 6.9, as a full run confirms). For $d \in \{3, 4\}$ **[computer]**: of the 771 597 groups P of order 3^8 , rank 3 or 4 and class 3 or 4, 3 986 pass the filter of Lemma 6.9; ANUPQ lists their 305 583 immediate descendants of step size 1; 64 362 have class 4, $Q'' \neq 1$ and cyclic centre, and 62 245 of these have $z \in Q'' \cap M_5$. For each of the 62 245 the truncated Magnus algebra gives $z \notin D_5$, exactly. The presentations given to it were checked independently with ANUPQ (order 3^9 , class 4, z central of order 3 in $Q'' \cap M_5$). The filter was validated against full expansion on a range and a random sample of 1500 parents, without loss of a candidate. The descendant step was repeated with an ANUPQ in which the defects found in an audit of the package are fixed (<https://ispacex.github.io/gap/>; two of them give wrong descendants for inputs not met here): for each of the 3 986 parents the numbers of descendants, of groups of class 4 with $Q'' \neq 1$ and

cyclic centre, and of candidates are the same, and the 62 245 presentations are identical. $\square$

Cost and data. The order-3⁹ run took 3 h 21 min on 4 cores; the filter cut the time per parent from about 350 to 27 ms. Code: `c38.g`, `c39.g` (C39Pass is Lemma 6.9), `c39all.g`, `d5z.py`, `chk38.g`, `r5.g` in `compute/magnus`.

7. OPEN QUESTIONS

- (1) Question 1.1; equivalently, is $D_5(Q) = 1$ for all finite 2-groups of class 3? **[open]** A counterexample must have $D_4(Q) \neq 1$. Using the truncated Magnus algebra $(\mathbb{Z}[G]/\Delta^n \cong T_{<n}(\mathbb{Z}^k)/I_n$ for a k -generator presentation, of rank $(k^n - 1)/(k - 1)$ independent of $|G|$) we checked the class-3 quotients of the three-relator Rips-type group of [BM23, §1.2] and of 13 variants (orders 2^{40} – 2^{107}): $D_5 = 1$ in all of them, including three, of orders 2^{84} , 2^{90} and 2^{107} , with $D_4 \neq 1$ **[computer; two independent implementations agree, and the exactness certificate holds in all cases, so $D_4 \neq 1$ and $D_5 = 1$ are proved, not only modulo 2^{30}]**. The method (code: <https://gitlab.com/ispacex/cflint-core>) was also checked against the lattice computation on 30 435 groups of order $\leq 3^7$ without discrepancies, and is 200–400 times faster at order 3^7 .
- (2) Is there 3-torsion in $D_5(G)/\gamma_5(G)$ for some G , i.e. is Tahara’s bound 6 attained? **[open]** (“Rips’s abyss”, Mikhailov’s 2013 talk.) Tahara expected the bound 6 to be the best one [Tah81, p. 142]. At the end of the paper he announced a separate paper [Tah81, p. 173], which would show $D_5(G) = \gamma_5(G) = 1$ for his 2-groups with $D_4 \neq \gamma_4$ [Tah77] and construct counterexamples to $D_5(G) = \gamma_5(G)$. The announcement does not say that their quotient D_5/γ_5 would have 3-torsion, and $D_5 \neq \gamma_5$ with 2-torsion is known anyway: $2 \mid s(5)$ in the notation of [BM23, §1]. We have not found that paper. It is not in zbMATH, nor in Tahara’s list of publications; his conference report of 1981 [THSH81] is on other topics; we have not seen his surveys in the Bulletin of Aichi University of Education (1981, 1982)[†]. His survey of 2003 [Tah03] reports Gupta’s claim that the exponent of D_n/γ_n is at most 2. Bartholdi and Mikhailov [BM23, §1] state the bound as $s(5) \in \{2, 6\}$, i.e. open. Known now: no 3-torsion when the Sylow 3-subgroup has order $\leq 3^9$, and $D_5 = \gamma_5$ when G/γ_2 is finite of exponent 3 (§6). A minimal 3-group with $D_5 \neq \gamma_5$ has order $\geq 3^{10}$, class 4, cyclic centre, and at least two invariants ≥ 9 in Q/γ_2 (Theorem 6.8). Open on paper: Q/γ_2 with two invariants ≥ 9 , e.g. $C_9 \times C_9 \times C_3$, where γ_2/γ_3 may have exponent 9, and an exact criterion for the nonterminal parents of type $(5, 2, 1)$ (Remark 6.11(3)).
- (3) The group analogue of Theorem C of [IMP26]: is $D_{2c-1}(G) = 1$ for every metabelian group of class $\leq c$? For $c = 3$ this is Question 1.1. **[open]** For p odd and $c \leq p+1$ the answer is yes by [GT85, Thm. 3.2]

(reducing to finite metabelian p -groups); open are $p = 2$, $c \geq 3$, and p odd, $c \geq p + 2$. The plateau group of [IMP26, §3.3] is metabelian of class 4 with $D_6 \neq 1$ **[computer]**, so $2c - 2$ is attained, as for Lie rings. Its distinguished element w is not in D_7 , and since $D_6 = \langle w \rangle$ [IMP26], $D_7 = 1$ **[computer; the dense Howell form modulo 2^{32} of the ideal of relations in the truncated Magnus algebra of rank 55 986, with the exactness certificate]**: this group is not a counterexample, and the analogue, if true, is sharp at $c = 4$. A first search for a counterexample at $c = 4$ reduces to metabelian quotients of class 4 with γ_4 finite cyclic, testing its element of order 2 **[paper; uses $D_5^2 \subseteq G''\gamma_5$ [MP09, Thm. 2.28]]**; about 170 such groups (quotients and perturbations of the plateau group, group versions of the Lie ring L_1 of [IMP26, Ex. 3.2], random presentations on 4 generators) all have $D_7 \cap \gamma_4 = 1$ **[computer]**. For $c = 5$, the plateau group shifted up one class (five generators) has $D_7 \neq 1$ but $D_8 \cap \gamma_5 = 1$ **[computer; certified]**, so for metabelian groups of class 5 at least $D_7 \neq 1$ is attained, one short of $2c - 2$; an elementary pairing lemma indicates that the construction would need the exponents of the relations of [IMP26, (3.12)] quadrupled rather than doubled at class 5 **[paper]**, and our attempts in that direction collapse the group **[computer]**. Code and data: <https://gitlab.com/ispacex/cflint-fast-core/docs/analysis.md>, §§6.5–6.7).

REFERENCES

- [BM23] L. Bartholdi, R. Mikhailov, *Group and Lie algebra filtrations and homotopy groups of spheres*, J. Topology **16** (2023), 822–853; arXiv:1805.10894.
- [BP15] L. Bartholdi, I. B. S. Passi, *Lie dimension subrings*, Internat. J. Algebra Comput. **25** (2015), 1301–1325.
- [GNO] G. Gamble, W. Nickel, E. A. O’Brien, *ANUPQ: ANU p -Quotient*, GAP package, version 3.3.2.
- [GK92] N. Gupta, Yu. V. Kuz’min, *On varietal quotients defined by ideals generated by Fox derivatives*, J. Pure Appl. Algebra **78** (1992), 165–172.
- [Gru57] K. W. Gruenberg, *Residual properties of infinite soluble groups*, Proc. London Math. Soc. (3) **7** (1957), 29–62.
- [GT85] N. Gupta, K.-I. Tahara, *Dimension and lower central subgroups of metabelian p -groups*, Nagoya Math. J. **100** (1985), 127–133; doi:10.1017/S0027763000000258.
- [Gup02] N. D. Gupta, *The dimension subgroup conjecture holds for odd order groups*, J. Group Theory **5** (2002), 481–491.
- [HMP07] M. Hartl, R. Mikhailov, I. B. S. Passi, *Dimension quotients*, J. Indian Math. Soc., Centenary Volume (2007), 63–107; arXiv:0803.3290.
- [IMP26] V. Ionin, R. Mikhailov, T. Petrov, *Dimension subrings of Lie rings*, arXiv:2610.02042 (2026); Lean code: <https://github.com/TimmPeterson/lie-algebras>.
- [Jen41] S. A. Jennings, *The structure of the group ring of a p -group over a modular field*, Trans. Amer. Math. Soc. **50** (1941), 175–185.
- [KM26] E. I. Khukhro, V. D. Mazurov (eds.), *Unsolved problems in group theory. The Kourovka notebook*, 21st ed., 2026; arXiv:1401.0300.

- [MP09] R. Mikhailov, I. B. S. Passi, *Lower central and dimension series of groups*, Lecture Notes in Math. **1952**, Springer, 2009.
- [OBr90] E. A. O'Brien, *The p -group generation algorithm*, J. Symbolic Comput. **9** (1990), 677–698.
- [Pas68] I. B. S. Passi, *Dimension subgroups*, J. Algebra **9** (1968), 152–182.
- [Rip72] E. Rips, *On the fourth integer dimension subgroup*, Israel J. Math. **12** (1972), 342–346.
- [Tah77] K.-I. Tahara, *On the structure of $Q_3(G)$ and the fourth dimension subgroups*, Japan. J. Math. (N.S.) **3** (1977), 381–394; doi:10.4099/math1924.3.381.
- [Tah81] K.-I. Tahara, *The augmentation quotients of group rings and the fifth dimension subgroups*, J. Algebra **71** (1981), 141–173; doi:10.1016/0021-8693(81)90111-3 (Elsevier open archive).
- [Tah03] K.-I. Tahara, *Survey on dimension subgroup problem*, in: Recent advances in group theory and low-dimensional topology (Pusan, 2000), Res. Exp. Math. **27**, Heldermann, Lemgo, 2003, 167–181[†] (seen only through its zbMATH review, Zbl 1055.20002).
- [THSH81] K.-I. Tahara, K. Horibe, M. Shimizu, A. Hosomi, *Some topics in integral group rings*, RIMS Kôkyûroku **438** (1981), 55–65.